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Abstract

This paper is aimed at introducing hypothesis testing, focusing on the paired t-test. It will explain how the paired t-test is applied to statistical analyses using an example. Specific formulas that are used to calculate values based on the data recorded in the example are given. This paper was originally submitted as part of the required senior Colloquium presentation for Mathematics majors at Xavier. It is required to research a topic in mathematics or statistics, and present it to fellow students, faculty, and staff of the mathematics department.

Key Terms:
Paired t-test; p-value; Student t-test; Hypothesis Testing

Introduction

Statistical analysis involves the calculation of the mean of a set of values in a sample used for observational study. Statistical analysis can be applied in many fields. There are now, many methods that are used to perform a statistical analysis. Hypothesis testing is one method used in statistics. The objective of this paper to explain a form of hypothesis testing, called the paired t-test.

Hypothesis Testing

Hypothesis testing is used to make an inference about a population that's under study. The inference is based on the parameter(s) for the statistic, usually the sample mean and standard deviation. Suppose it is believed that the mean of a population is zero, the first step in hypothesis testing is to state the null hypothesis (H_0) and an alternative hypothesis (H_a). The null hypothesis is the assumption that the mean will be equal to zero. The alternative hypothesis is the assumption that the mean will be either greater than zero, less than zero, or simply, not equal to zero. When the alternative hypothesis states that the mean is less than zero, the test is called a left-tailed test. It is right tailed when H_a states that the mean is greater than zero. The test is called two-tailed when H_a states that the mean is not equal to zero. The second step in hypothesis testing is to calculate a test statistic. Based on the value of the test statistic, you will find the p-value from the table of z-distributions, which is based upon the normal distribution known as the bell curve. Normal distribution is a function expressed as $f(x) = \frac{-1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, where

μ is the mean and σ is the standard deviation of a population. It is displayed as a bell-shaped curve about its mean. From the p-value you will make a decision to either reject or not reject the null hypothesis.

The determined p-value is compared to the level of significance, α (alpha). The level of significance is the probability of making a Type I error. A Type I error occurs when H_0 is rejected when it is true. If H_0 is not rejected when it is actually false, the error is called as a Type II error. The alpha level can be computed as one minus the confidence level. The confidence level is equivalent to the area under the curve. Common values for the confidence level are .90, .95, and .99. Of the three, the most commonly used is the confidence level at .95. This suggests a 95% confidence that the null hypothesis is true. Thus, the level of significance that is commonly used is .05(1 - .95). Thus, α set at .05 suggests that there is a 5% chance of making a Type I error. If the p-value is less than or equal to α, the null hypothesis should be rejected. If the p-value is greater than α, H_0 should not be rejected. The last step is to explain the results of the test and what it concludes about the analysis.

Paired t-test

The paired t-test is a type of hypothesis testing that is used when two sets of data are being observed. The data in a paired t-test are dependent, because each value in the first sample is paired with a value in the second sample. The parameter used to make the inference is the difference of the means of both data sets. Similar to our previous hypothesis test example, the null hypothesis states that the difference of the means is equal to zero $(\mu_1 - \mu_2 = 0)$. This can also be understood as the means are equal. The alternative hypothesis can be the mean of the first sample is greater than the mean of the second $(\mu_1 - \mu_2 > 0)$, the mean of the first sample is less than the mean of the second sample $(\mu_1 - \mu_2 < 0)$, or the means are not equal with no greater than or less than comparison $(\mu_1 - \mu_2 \neq 0)$. Thus,

Example 1:

$$H_0 \ : \ \mu_1 = \mu_2$$

$$H_a \ : \ \mu_1 \neq \mu_2$$

Example 2:

$$H_0 \ : \ \mu_1 = \mu_2$$

$$H_a \ : \ \mu_1 > \mu_2$$

$$\mu_1 < \mu_2$$

Student's t-test

The Student's t-test is the test statistic used in a paired t-test. The Student's t is a distribution discovered by a statistician, W.S. Gosset. Because he worked for a company that did not approve of its employees publishing their research, when publishing his work he used the name Student. When a population is too large for testing, samples are taken from the population and used in a test. Sampling is expensive and the t-distribution is ideal when the sample is small (i.e. < 30). The t-distribution was created for use when σ is unknown. The student t-distribution is similar to normal distribution in that it is bell shaped and symmetrical. Its shape depends on the degrees of freedom, which is one less than the sample size ($n-1$). As degrees of freedom increase, the t-distribution approaches normal distribution. Gosset developed the table of t-distributions to find the p-value that corresponds to the test statistic and the degrees of freedom. The values in the Student t-table are calculated by formulas developed by Gosset. This falls under another subject called mathematical statistics, and is beyond the scope of this scholarly note.

Formula to calculate the t test statistic:

$$d = x_1 - x_2 \qquad \text{difference}$$

$$\bar{d} = \frac{\sum d}{n} \qquad \text{mean difference}$$

$$s_d = \sqrt{\frac{\sum\left(d-\bar{d}\right)^2}{n-1}} \qquad \text{sample standard deviation}$$

$$t = \frac{\bar{d}}{s_d} \cdot \sqrt{n} \qquad \text{t test statistic}$$

Demonstrating the Paired t-test

The following sample problem will be used to show how to apply the paired t-test. The purpose of the test is to determine whether a person's physical condition improves after jogging. An investigator obtains maximal VO_2 before subjects start jogging and again six months later. The first sample contains the values of VO_2 that was recorded for each participant before they started jogging. The second sample contains the values of VO_2 recorded after jogging.

The data used is given in the following display:

	Before	After	Difference	Diffsqrd
1	34.1	47.9	-13.8	1.9488
2	32.3	44.6	-12.3	.0108
3	36.5	47.3	-10.8	2.5728
4	38.6	50.6	-12.0	.1632
5	39.6	51.9	-12.3	.0108
6	31.8	43.3	-11.5	.8172
7	31.0	43.3	-12.3	.0108
8	38.8	51.9	-13.1	.4844
9	29.3	41.2	-11.9	.2540
10	35.3	47.6	-12.3	.0108
11	41.3	54.0	-12.7	.0876
12	43.3	55.6	-12.3	.0108
13	33.8	45.6	-11.8	.3648
14	28.3	39.4	-11.1	1.7004
15	36.8	48.9	-12.1	.0924
16	30.6	42.4	-11.8	.3648
17	28.8	46.3	-17.5	25.9692
18	40.4	52.8	-12.4	.0000
19	39.8	48.9	-9.1	10.9164
20	44.8	56.7	-11.9	.2540
21	30.8	46.5	-15.7	10.8636
22	25.8	38.7	-12.9	.2460
23	32.7	44.2	-11.5	.8172
24	35.3	47.2	-11.9	.2540
25	37.9	51.0	-13.1	.4844

(In this table, Diffsqrd represents $(d-\overline{d})^2$.)

$$\sum d = -310.1$$

$$\overline{d} = \frac{-310.1}{25} = -12.404$$

$$\sum (d-\overline{d})^2 \approx 60.86$$

$$s_d = \sqrt{\frac{60.86}{24}} \approx 1.59$$

Step 1: H_0: The physical condition is the same after jogging
$(\mu_1 = \mu_2)$ $\alpha = 0.05$

H_a : The physical condition improves after jogging

$(\mu_1 \neq \mu_2)$ This is a two tailed test.

Step 2: $t = \frac{-12.404}{1.59}\sqrt{25} \approx -39$

Step 3: Since -39 is not on the t-distribution table, the p-value is approximately zero.

Step 4: Since $0 < 0.05$, reject the H_0 in favor of the alternative hypothesis.

Step 5: Because H_0 was rejected, we can conclude that jogging does improve a person's physical condition.

Conclusion

This scholarly note has presented a brief overview of hypothesis testing and focused on understanding the paired t-test. An example using the paired t-test was given developing the null and alternative hypothesis, calculating the test statistics, and drawing an inference based upon the level of confidence desired. For additional information regarding any of the topics covered herein, the reader should refer to the books in the bibliography.

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